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S-Matrix Identities in Affine Toda Field Theories

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Abstract

We note that S-matrix/conserved charge identities in affine Toda field theories of the type recently noted by Khastgir can be put on a more systematic footing. This makes use of a result first found by Ravanini, Tateo and Valleriani for theories based on the simply-laced Lie algebras (A, D and E) which we extend to the nonsimply-laced case. We also present the generalisation to nonsimply-laced cases of the observation - for simply-laced situations - that the conserved charges form components of the eigenvectors of the Cartan matrix.

1 Introduction

Affine Toda field theory (ATFT) is a massive scalar field theory in 1+1 dimensions, with a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_a\partial_\mu\phi_a - \frac{m^2}{\beta^2}\sum_{j=0}^{r} n_j \exp(\beta\alpha_j\cdot\phi). \tag{1}$$

The α_i, $j = 1, \ldots, r$ are the simple roots of a rank-r semi-simple Lie algebra g, α_0 being the affine root. There exists an ATFT related to each possible (twisted or untwisted) Dynkin diagram; quantum scattering matrices (S-matrices) were first found for the simply-laced cases [1, 2, 3], and later for nonsimply-laced cases [4, 5]. For theories based on nonself-dual algebras, the algebras fall naturally into dual pairs $(b_n^{(1)}, a_{2n-1}^{(2)}), (c_n^{(1)}, d_{n+1}^{(2)}), (g_2^{(1)}, d_4^{(3)})$, and $(f_4^{(1)}, e_6^{(2)})$. The S-matrix for the second member of each pair is related to that of the first by the strong/weak coupling duality $\beta \to \frac{4\pi}{\beta}$. The simply-laced algebras and $a_{2n}^{(2)}$ are self-dual.

In a recent paper, Khastgir [10] noted that there existed product identities on elements of these S-matrices, all evaluated at the same rapidity, associated with sum rules for the conserved charges of the theory. For example, he noted that in $a_r^{(1)}, d_r^{(1)}, (c_r^{(1)}, d_{r+1}^{(2)})$ and $(b_r^{(1)}, a_{2r-1}^{(2)})$ we had

$$S_{22}(\theta) = S_{11}(\theta)S_{13}(\theta), \qquad q_s^2 q_s^2 = q_s^1 q_s^1 + q_s^1 q_s^3, \tag{2}$$

where $S_{ab}(\theta)$ is an element of the S-matrix evaluated at rapidity θ, and q_s^x is the xth conserved charge at spin s. The above is true for any s, with non-trivial independent conserved charges existing for s an exponent of the Lie algebra. This is a consequence of the fact - first noted in [2] (see also [12]) - that the coefficients of a Fourier series expansion of the logarithmic derivative of the S-matrix provide solutions to the conserved-charge bootstrap. If we define

$$-i\frac{d}{d\theta}\ln S_{ab}(\theta) = \varphi_{ab}(\theta) = -\sum_{k=1}^{\infty}\varphi_{ab}^{(k)}\mathrm{e}^{-k|\theta|}, \tag{3}$$

then the linearly independent rows and columns of $\varphi^{(k)}$ each satisfy the conserved charge bootstrap, and so are proportional to the unique vector q_k, implying $\varphi_{ab}^{(k)} \propto q_k^a q_k^b$. This does allow the possibility that some of the matrices $\varphi^{(k)}$ are zero, making their relationship to the conserved charges trivial, but we shall see later that this does not happen, at least for generic couplings.

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Khastgir did not give a general prescription as to how to find such S-matrix identities, beyond noting that some can be written down by inspecting the product-form S-matrix. We give such a prescription, and also expand the list of identities to cases when not all the S-matrices are evaluated at the same rapidity.

In order to find these identities, we need to make use of an identity found by Ravanini, Tateo and Valleriani (RTV) [14] for simply-laced cases, and to generalise it to nonsimply-laced cases. We establish this fundamental identity in Section 2 before using it, in Section 3, to generate further S-matrix identities.

2 The Basic Identities

In simply-laced cases, Ravanini, Tateo and Valleriani [14] found

$$S_{ab}\left(\theta+\frac{i\pi}{h}\right)S_{ab}\left(\theta-\frac{i\pi}{h}\right)=\mathrm{e}^{-2i\pi G_{ab}\Theta(\theta)}\prod_{c=1}^{r}S_{ac}(\theta)^{G_{bc}}, \tag{4}$$

for the minimal S-matrix and gave a general proof based on the geometric formulae of [6]. This identity also holds for the full S-matrix. Here, G is the incidence matrix of the Dynkin diagram of the Lie algebra, h is the Coxeter number, and Θ is the step function

$$\Theta(x)=\lim_{\epsilon\to 0}\left[\frac{1}{2}+\frac{1}{\pi}\arctan\frac{x}{\epsilon}\right]=\begin{cases}0 & \text{if} \quad x<0,\\ \frac{1}{2} & \text{if} \quad x=0,\\ 1 & \text{if} \quad x>0.\end{cases} \tag{5}$$

There also exist integral formulae for the S-matrix, given by Oota [13] and Frenkel and Reshetikhin [8]; these can be used to prove RTV's formula but, since they apply to *all* simple Lie algebras, can also be used to generalise (4) to nonsimply-laced cases.

Before we can present Oota's formula, we must break off to provide a few definitions. We shall work with the untwisted algebras here, since the S-matrices for the theories based on twisted algebras can be found through the duality transformation. (The exceptional case $a_{2n}^{(2)}$ will be discussed at the end of this section.) Firstly, let $h^\vee$ be the dual Coxeter number, $r^\vee$ be the maximum number of edges connecting any two vertices of the Dynkin diagram†, and $0\le B\le 2$ be a function of the coupling constant, conjectured to be [1, 2, 3, 7]

$$B^g=\frac{2\beta^2}{\beta^2+\frac{4\pi h}{h^\vee}} \tag{6}$$

for Lie algebra g. Next, let the set of simple roots of the Lie algebra be $\{\alpha_i\}$, and define $t_i=r^\vee\frac{(\alpha_i,\alpha_i)}{2}$, where the length of the long roots is normalised to $2^\ddagger$. Finally, define

$$[K_{ij}]_{q\overline{q}}=(q\overline{q}^{t_i}+q^{-1}\overline{q}^{-t_i})\delta_{ij}-[G_{ij}]_{\overline{q}}, \tag{7}$$

and

$$M_{ij}(q,\overline{q})=([K]_{q\overline{q}})^{-1}_{ij}\,[t_j]_{\overline{q}}, \tag{8}$$

where $q(t)=\exp\left(\frac{(2-B)t}{2h}\right)$, $\overline{q}(t)=\exp\left(\frac{Bt}{2r^\vee h^\vee}\right)$ and we use the standard notation $[n]_q=(q^n-q^{-n})/(q-q^{-1})$. Oota found that

$$S_{ab}(\theta)=(-1)^{\delta_{ab}}\exp\left(4\int_{-\infty}^{\infty}\frac{dk}{k}\mathrm{e}^{ik\theta}\left\{\sin k\theta_h\cdot\sin k\theta_H\cdot M_{ab}(q(\pi k),\overline{q}(\pi k))+\frac{\delta_{ab}}{4}\right\}\right), \tag{9}$$

where, for conciseness, we have defined $\theta_h=\frac{i\pi(2-B)}{2h}$ and $\theta_H=\frac{i\pi B}{2r^\vee h^\vee}$. The formula given by Frenkel and Reshetikhin [8] is similar to this, but without the factor of $(-1)^{\delta_{ab}}\exp\left(\int_{-\infty}^{\infty}\frac{dk}{k}\mathrm{e}^{ik\theta}\delta_{ab}\right)$. The

†This is 1 for the self-dual cases, and 2 for the nonself-dual ones, except for $(g_2^{(1)},d_4^{(3)})$ where it is 3.

‡Thus $t_i=1$ for short roots and $t_i=r^\vee$ for long roots.

standard Fourier transform result $\int_{-\infty}^{\infty}\frac{dk}{k}\mathrm{e}^{ik\theta} = i\pi\mathrm{sgn}\,\theta$, together with the $2i\pi$ periodicity of the exponential, shows that this factor is 1 for θ real, though it is different from 1 if θ is complex. We shall use Frenkel and Reshetikhin's form here.

We now propose a generalised RTV identity of the form

$$S_{ij}(\theta+\theta_h+t_i\theta_H)S_{ij}(\theta-\theta_h-t_i\theta_H) = \mathrm{e}^y \prod_{l=1}^{r}\prod_{n=1}^{G_{il}} S_{jl}(\theta+(2n-1-G_{il})\theta_H), \tag{10}$$

and aim to find y. We now need to be careful, particularly for $\theta=0$, as in this case we can either consider the lhs as

$$\lim_{X\to\theta_h+t_i\theta_H}\lim_{\theta\to 0} S_{ij}(\theta+X)S_{ij}(\theta-X) \tag{11}$$

or with the limits reversed. In the first case (the one we would choose if we were to go on and use this in the thermodynamic Bethe ansatz approach) it simply becomes the unitarity constraint on S, and thus is equal to 1. If we take the limits in the other order, then (as can be seen if the S-matrix is written out in block form) the result becomes -1 if the S-matrix has a pole at $\theta_h+t_i\theta_H$. Similar situations can also arise for the rhs, and in any other case where we have a product $S(\theta+X)S(\theta-X)$ such that one term is evaluated at a pole of the S-matrix, and the other is at a zero.

Following RTV, we will consider the first case here. We found it simplest to replace θ_h and θ_H in (10) by $\theta_h+i\epsilon$ and $\theta_H+i\epsilon$, and take the limit $\epsilon\to 0$ last. Substituting in (9) and simplifying, we find

$$\begin{aligned}\mathrm{e}^y = \lim_{\epsilon\to 0}\exp\Bigg(\sum_{l=1}^{r}\int_{-\infty}^{\infty}\frac{dk}{k}\mathrm{e}^{ik\theta}\cdot \\ \{q(\pi k)-q(-\pi k)\}\{\overline{q}(\pi k)-\overline{q}(-\pi k)\}[K_{il}]_{q'(\pi k)\overline{q}'(\pi k)}M_{lj}(q(\pi k),\overline{q}(\pi k))\Bigg)\end{aligned} \tag{12}$$

where $q'(t)=q(t)\mathrm{e}^{\frac{\epsilon t}{\pi}}$ and $\overline{q}'(t)=\overline{q}(t)\mathrm{e}^{\frac{\epsilon t}{\pi}}$. Looking ahead to Section 4 and formula (23), we can see that when the integrand in (12) is expanded out, all the terms are of the form $t(x,\theta)=\int_{-\infty}^{\infty}\frac{dk}{k}\mathrm{e}^{ik\theta}\mathrm{e}^{x|k|}$, with x real, which is divergent if x is positive. It is, however, implicit in Oota's formulation that any terms which are naïvely divergent must be analytically continued. For x negative, $t(x,\theta)$ is just another standard Fourier transform, which has the result $-2i\arctan\frac{\theta}{x}$, so, after analytic continuation, we should set $t(x,\theta)=-2i\arctan\frac{\theta}{x}$ for all x.

It is helpful to split the $t(x,\theta)$ terms into two sets: $x\to 0$ in the limit $\epsilon\to 0$ (but non-zero otherwise) and the rest. It is only for the first of these that the choice of limit prescription makes a difference. Taking the ϵ limit first, they are $\pm i\pi$, depending on whether $x\to 0\mp$, for all θ, but, with the limits taken the other way, we get $\lim_{x\to 0}-2i\arctan\frac{\theta}{x}$.

If we had chosen to take the ϵ - rather than the θ - limit first, $\sum_{l=1}^{r}[K_{il}]_{q\overline{q}}M_{lj}(q(\pi k),\overline{q}(\pi k))$ would have reduced to $\delta_{ij}[t_j]_{\overline{q}(\pi k)}$. Each $t(x,\theta)$ would then be matched by a $t(-x,\theta)$, so the rhs would reduce to 1, i.e. $y=0$. Looking carefully at the above reasoning, we can convert this to a result about the other limit prescription by adding $-2i\pi\Theta(\theta)$ to y for each term of the first type present, this being the difference between its value with the ϵ limit taken first - which we take to be $i\pi$[§] - and its value with the θ limit taken first.

If we take first the term $l=i$ then, looking at the definition (7), $t(x,\theta)$ terms of the first type are only present if $M_{ij}(q(\pi k),\overline{q}(\pi k)$ is of the form $q(\pi|k|)^{-2}\overline{q}(\pi|k|)^{-t_i-1}+$(terms in more negative powers of $q,\overline{q}$). Expanding out equation (23), we find M is of the form $q^{-x}\overline{q}^{-y}+$ more negative powers for given x,y, meaning we are searching for the presence of a block $\{2,t_i+1\}$ in $S_{ij}(\theta)$. This is no surprise, as this is the block responsible for the pole $\theta_h+t_i\theta_H$ discussed above; this is just another way of representing the same situation. Case-by-case analysis shows this block is present iff G_{ij} is odd.

Looking instead at $l\neq i$, we note that then $[K_{il}]_{q\overline{q}}=-[G_{il}]_{\overline{q}}$ which, expanding out the definition of the bracket notation, is $-\overline{q}^{G_{il}-1}-\overline{q}^{G_{il}-3}-\ldots-\overline{q}^{-(G_{il}-1)}$. Thus, the only way a $\overline{q}'$ dependence can enter is if $G_{il}>1$. In this case, by an argument similar to the above, we are now searching for

[§]In this context, due to the $2i\pi$ periodicity of the exponential, we need not worry about how the $x\to 0$ limit is taken.

$\{1, t_i\}$ blocks, with $t_i > 1$. Proceeding again by case-by-case analysis, we find that the block $\{1, 2\}$ is never present and $\{1, 3\}$ is only possible in $(g_2^{(1)}, d_4^{(3)})$, where it does not occur in the right element of the S-matrix to invoke this process.

To sum up, we find a single contribution to y for G_{ij} odd and none otherwise. This can be restated as $y = -2i\pi\Theta(\theta)G_{ij}$, showing that we have indeed found a generalisation of the RTV formula.

To complete this section, we must discuss the exceptional case $a_{2n}^{(2)}$. Being self-dual, the S-matrix for this theory cannot be found from the above. Following Oota, however, we note that the necessary prescription is to replace each reference to $r^\vee h^\vee$ by $h^\vee = h = 2n + 1$, take all $t_a = 1$, and replace the incidence matrix by the "generalised incidence matrix" [11], which is obtained from the incidence matrix of $a_n^{(1)}$ by replacing the last zero on the diagonal by a one. Doing this, we obtain the correct integral S-matrix, and hence a generalised RTV identity, for this case.

3 Multi-linear Identities

The RTV result and its generalisation allow us to perform a simple trick and generate a large number of S-matrix identities. Interchanging i and j in (10) does not change the lhs if $t_i = t_j$ - the two roots are the same length - due to the symmetry of the S-matrix, so we can equate the rhs before and after interchanging to get

$$\prod_{l=1}^{r}\prod_{n=1}^{G_{il}} S_{jl}(\theta + (2n - 1 - G_{il})\theta_H) = \prod_{l'=1}^{r}\prod_{n'=1}^{G_{jl'}} S_{il'}(\theta + (2n' - 1 - G_{jl'})\theta_H). \tag{13}$$

(Note that the presence or absence of an exponential factor does not affect this, as $t_i = t_j$ ensures $G_{ij} = G_{ji}$.) If i and j are such that the corresponding rows of the incidence matrix consist of entries no greater than 1, this reduces to

$$\prod_{l=1}^{r} S_{il}(\theta)^{G_{lj}} = \prod_{l'=1}^{r} S_{jl'}(\theta)^{G_{l'i}}, \tag{14}$$

and we can obtain identities for products of S-matrix elements, all evaluated at the same rapidity; the first example, (2), is one of this set. Now, however, we also have identities in which not all rapidities are equal.

To generalise the connection between S-matrix product identities and conserved charge sum rules to this case, we can take logarithmic derivatives to find that if

$$\prod_{a,b\in\{i,j\}} S_{ab}(\theta + if^1_{ab}) = \prod_{a',b'\in\{i',j'\}} S_{a'b'}(\theta + if^2_{a'b'}), \tag{15}$$

for some sets $\{i, j\}$ and $\{i', j'\}$ then

$$\sum_{a,b\in\{i,j\}} \mathrm{e}^{-if^1_{ab}s} q_s^a q_s^b = \sum_{a',b'\in\{i',j'\}} \mathrm{e}^{-if^2_{a'b'}s} q_s^{a'} q_s^{b'}. \tag{16}$$

Applying this to (13) gives

$$\sum_{l=1}^{r}[G_{il}]_{\overline{q}(i\pi s)} q_s^l q_s^j = \sum_{l'=1}^{r}[G_{jl'}]_{\overline{q}(i\pi s)} q_s^{l'} q_s^i, \tag{17}$$

where it should be noted that the sums over n and n' in (13) have been absorbed by the introduction of the $[G_{ab}]_{\overline{q}(\pi s)}$ notation.

To give a simple example of this result, in the $b_r^{(1)}$ algebra we have, for $1 < i < r - 1$

$$S_{(r-1)(i-1)}(\theta)S_{(r-1)(i+1)}(\theta) = S_{i(r-2)}(\theta)S_{ir}(\theta + \theta_H)S_{ir}(\theta - \theta_H), \tag{18}$$

and

$$q_s^{r-1}q_s^{i-1} + q_s^{r-1}q_s^{i+1} = q_s^i q_s^{r-2} + \frac{1}{2}\cos\frac{B^{b_r^{(1)}}\pi s}{2r^\vee h^\vee} \cdot q_s^i q_s^r, \tag{19}$$

with (through the duality transformation $B^{b_r^{(1)}} \to 2 - B^{a_{2r-1}^{(2)}}$) corresponding identities for $a_{2r-1}^{(2)}$.

4 Other Observations

In the simply-laced cases, the conserved charges can be simply characterised as components of the eigenvectors of the Cartan matrix [11]. Taking the logarithmic derivative of the formula (10) and proceeding as before allows us to generalise this to nonsimply-laced cases, as follows:

$$\sum_{l=1}^{r}[G_{il}]_{\overline{q}(i\pi s)}q_s^l = 2\cos\left[\pi s\left(\frac{2-B}{2h}+\frac{Bt_i}{2r^\vee h^\vee}\right)\right]q_s^i. \tag{20}$$

Note now, however, that the t_i in the cos term stops this from being a proper eigenvalue equation in the nonsimply-laced cases. In simply-laced cases, this reduces to the known eigenvector result, since $[n]_q = n$ for $n = 0, 1$ (as all entries of the incidence matrix are in these cases), and we have all $t_i = 1$ and $h = r^\vee h^\vee$. Rearranging, this can also be stated as

$$\sum_{l=1}^{r}[K_{il}]_{q(i\pi s)\overline{q}(i\pi s)}q_s^l = 0. \tag{21}$$

We can also find a relation between the matrix M and the conserved charges. Noting that the S-matrix expression (9) explicitly contains the matrix M, we first take the logarithmic derivative, and then note that the resulting integral can be re-expressed as a contour integral over the upper half-plane. The only poles in this expression are in the matrix M, so, before we can continue, we must find a formula for this. If we recall that the S-matrix can be written in a product form [7] as

$$S_{ab}(\theta) = \prod_{x=1}^{h}\prod_{y=1}^{r^\vee h^\vee}\{x,y\}^{m_{ab}(x,y)} \tag{22}$$

(where the $\{x, y\}$ are standard building blocks, and the $m_{ab}(x, y)$s are integers), and compare with Oota's integral form, we can find a formula for M as

$$M_{ab}(q,\overline{q}) = \sum_{x=1}^{h}\sum_{y=1}^{r^\vee h^\vee} m_{ab}(x,y)\frac{q^{h-x}\overline{q}^{r^\vee h^\vee-y} - q^{-(h-x)}\overline{q}^{-(r^\vee h^\vee-y)}}{q^h\overline{q}^{r^\vee h^\vee} - q^{-h}\overline{q}^{-r^\vee h^\vee}}. \tag{23}$$

This shows that the only poles present are at $k = im$, m being any integer, so the result is that we can re-express the integral in the form of a Fourier expansion, and thus read off a relation between $\varphi_{ab}^{(s)}$ and M as

$$\varphi_{ab}^{(s)} = 2\sin\pi s\cdot\sinh s\theta_h\cdot\sinh s\theta_H\cdot M(q(i\pi s),\overline{q}(i\pi s)). \tag{24}$$

Of course, to find an expression in $q_s^a q_s^b$, we need to include a scaling factor. Noting that $\sum_{i=1}^{r} q_{s_i}^a q_{s_i}^b = \delta_{ab}$, where s_i is the ith exponent of a rank-r algebra, we could use $q_s^a q_s^b = \varphi_{ab}^{(s)} / \sum_{i=1}^{r}\varphi_{11}^{(s_i)}$.

Combining this with the expression for M, we get

$$\varphi_{ab}^{(s)} = 2\sinh s\theta_h\cdot\sinh s\theta_H\cdot\sum_{x=1}^{h}\sum_{y=1}^{r^\vee h^\vee} m_{ab}(x,y)\sin\left(\frac{s\pi}{2}\left[\frac{(2-B)x}{h}+\frac{By}{r^\vee h^\vee}\right]\right). \tag{25}$$

From this, it is straightforward to see that the matrix $\varphi_{ab}^{(s)}$ is non-zero for generic B by simple case-by-case analysis. (This is different from this minimal case where, as noted by Klassen and Melzer [11], we can get a zero matrix for $s = \frac{h}{2}$ in simply-laced cases, even if that exponent is present.) Had there been cases where $\varphi^{(s)}$ was zero for some s, then taking the logarithmic derivative of an S-matrix identity would sometimes have resulted in a trivial conserved charge identity. As it is, however, we can always derive a non-trivial conserved charge identity from an S-matrix identity and vice versa.

5 Conclusions

We have found a generalisation of the RTV identity to nonsimply-laced cases, and, from this, a way of generating S-matrix identities of the type recently discussed by Khastgir. It is still an open question as to whether we have found all such identities, or merely a subset. In addition, we note that we can always use the technique of taking logarithmic derivatives to generate corresponding identities for the conserved charges. We have also, in equation (20), generalised the "eigenvector" characterisation of the conserved charges in the ATFTs to the nonsimply-laced cases.

While this note was in preparation, the result (10) was also reported by Fring, Korff and Schulz [9] as their "combined bootstrap" identity (we have altered our notation to accord with theirs). They proceeded by a geometrical argument, and went on to use this to derive the integral formula for the S-matrix. They chose to take the θ limit last, and so their bootstrap identity does not have the factor $e^{-2i\pi G_{ij}\Theta(\theta)}$, but it is otherwise the same. The result that $S_{ij}(\theta)$ contains the block $\{2, t_i+1\}$ iff G_{ij} is odd, which we found case-by-case, can also be found through a more systematic framework given by them.

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